Some hints on reading Boolos, “To Be is to Be. . . ”

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There are some things in this article that will only make sense to you if you’ve studied some metalogic and set theory. Don’t worry about that. We’ll be focusing on aspects of the article that should be intelligible to anyone who has taken 12A. That said, it’s not an easy article. Here are a few hints to get you through it.

On pp. 55–62, Boolos goes through a number of examples of English sentences that can only be symbolized using *second-order* quantifiers: quantifiers that bind variables in predicate position. You should make sure you understand the second-order formalizations Boolos gives and how they relate to the English sentences. You should also convince yourself that the sentences in question can’t be given first-order formalizations, by trying to give one.

On p. 57, Boolos offers a proof (due to David Kaplan) that (B) cannot be given a first-order formulation. The proof uses some concepts from metalogic, so don’t worry if you can’t understand it. For those who are interested, though, here’s the basic idea:

1. If there were a first-order formula that captured the meaning of (B), it would be possible to give first-order axioms for arithmetic that rule out nonstandard models.
2. But it can be proven that no first-order axioms for arithmetic can rule out nonstandard models. (This follows fairly directly from the *compactness theorem* for first-order logic.)
3. Hence (by reductio) there is no first-order formula that captures the meaning of (B).

What’s a “nonstandard model” of arithmetic? Well, you know what a *model* is (a domain and an interpretation of the language’s predicates and individual constants on that domain). A model of a set of axioms is a model that makes these axioms true. Now consider a set of first-order axioms for arithmetic (such as the standard Peano axioms). These axioms will contain some arithmetical expressions, like ‘0’, ‘S’, ‘+’, and ‘<’. The *standard model* of arithmetic interprets these in the obvious way: the domain is the set of natural numbers, the extension of ‘S’ is the set of pairs consisting of a natural number and its successor, the extension of ‘+’ is the set of triples consisting of two natural numbers and their sum, and the extension of ‘<’ is the set of pairs consisting of two natural numbers where the first is less than the second. But the standard model is not the only model of the axioms. There are *nonstandard models* whose domains contain lots of “extra numbers” that are greater than all the standard natural numbers. You might think we could rule out these models by adding more axioms, but it can be proven that no matter how many new first-order axioms we add, there are still going to be nonstandard models. (The proof is neat, and fairly simple once you’ve got compactness. There’s a quick sketch in the Wikipedia entry for Non-standard arithmetic.)

Kaplan establishes premise (1) of his argument by giving a sentence (C) that is a substitution instance of (B), with ‘$(x = 0 \vee x = y + 1)$’ put in for ‘Axy’. Clearly, if there is a first-order representation of (B), there will be a first-order representation of (C). He then shows that (C) is true in every nonstandard model of arithmetic, but false in the standard model (see note 7 for the details). Thus, if there were a first-order formula equivalent to (C), we would have a first-order way to rule out all nonstandard models of arithmetic (just add the negation of (C) to the other axioms). Since it can be proven on general grounds that there is no way to do this, we know there can't be a first-order formula equivalent to (C).

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Note that (L) on p. 59 is a complicated way of analyzing the relation: x is an ancestor of y. Think about how this works.

Boolos says that (K) is a way of saying "x is a standard natural number" (that is, not one of the "extra numbers" that you get in nonstandard models). The idea is that the standard numbers are the ones you will eventually reach if you start from 0 and follow the "successor" relation up (0, 1, 2, ...). If (K) were first-orderizable, then we'd have a first-order way to exclude nonstandard models.

On p. 60 Boolos gives an example of a valid *inference* that cannot be represented in first-order logic. It's a bit convoluted, but it's fairly simple once you understand (L).

Be sure you understand why Quine's first-order analysis of (P) does not adequately capture its meaning (at least on one reading).

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On pp. 62–64, Boolos looks at some sentences that do have first-order equivalents, but are *more naturally* represented by second-order formulas.

On p. 64, Boolos begins to examine some classic sentences of set theory. Here '$x \in y$' means "x is a member of y." Throughout this section, Boolos assumes that the quantifiers range over sets, so '$\exists x$' gets translated "there is a set." Don't be confused by that.

If you haven't ever studied set theory, some of the things Boolos talks about will be opaque. (For example, you might wonder, what the heck is *Aussonderung*? It's a standard axiom of set theory.) Don't worry about that. You don't need to know much to get the main points Boolos is making.

One important thing you *should* know about is *Russell's Paradox*. It's natural to think that, for any condition you can dream up, there's a set of the objects that satisfy that condition. For example, there's a set of things that satisfy 'x is a dog', and a set of things that satisfy 'x is a natural number divisible by 2.' Bertrand Russell showed that this natural principle leads to paradox. Consider the condition, 'x is not a member of itself', that is, '$\neg(x \in x)$'. According to our naive principle, there should be a set of things that satisfy this condition: a set of things that are not members of themselves. Call it R. Is R a member of itself? Well, a little reflection will convince you that if it is, then it isn't, and if it isn't, it is. Presumably, then, there can be no such set, and our naive principle must be rejected.

Boolos's (d) on p. 64 looks like a second-order representation of Russell's Paradox, and (e) looks like a second-order representation of the naive principle that for any condition there's a set of things that satisfies it. And that's just what they would be, if '$\exists X$' were understood as quantifying over *sets*, as many critics of second-order logic thought it must be. (Quine called second-order logic "set theory in sheep's clothing." See his *Philosophy of Logic*.) That's why Boolos asks, on top of p. 65: "Are we not here on the brink of a well-known abyss?"

One way out of the abyss is to restrict the first-order quantifiers to *sets*, and let the second-order quantifiers range over subsets of the domain of the first-order quantifiers. This blocks the paradox, but only at the cost of making it impossible for us to talk about *all sets*. (In standard set theories, there is no set of all sets.)

Here Boolos wants to take another approach, which he articulates beginning on p. 66. Instead of interpreting the second-order quantifiers as ranging over sets (or classes or collections) of things, he gives us a way of translating second-order formulas into sentences of English involving *plural* quantification. The crucial part of the translation scheme he proposes is the first full paragraph of p. 68. The complications here stem from the fact that we want to be able to say, for example, '$\exists X \forall x (Xx \equiv x \neq x)$', and we can't render this as "there are some things all of which are non-identical." See if you can understand Boolos's example on p. 68.

Boolos gives his punch line in the second full paragraph on p. 72. Read this carefully.