Generalized Quantifiers and Definite Descriptions

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1 What is a quantifier?

Webster's *Ninth New Collegiate Dictionary* gives two definitions:

> a. a prefixed operator that binds the variables in a logical formula by specifying their quantity. b. a limiting noun modifier (as *five* in "the five young men") expressive of quantity and characterized by occurrence before the descriptive adjectives in a noun phrase.

Let's start by thinking about quantifiers in natural language. Using definition (b) as a paradigm, we can generate some clear examples:

1. *five* young men
2. *no* young men
3. *all* young men
4. *some* young men
5. *two* young men
6. *more than six* young men
7. *at most four* young men
8. *a few* young men
9. *many* young men
10. *most* young men
11. *a* young man

Notice that, grammatically, these quantifiers are *determiners*: they modify a noun phrase. (Other determiners in English include *the*, *my* and *those*.) To form a sentence using a quantifier, one generally needs to add two things: a (possibly modified) noun (obtaining *five young men*) and a verb phrase (obtaining *Five young men sang in harmony*).[1]

In this respect, our natural-language quantifiers are different from the familiar quantifiers of first-order logic, which just require you to add *one* thing (an open formula) to get a sentence. We can better capture the grammatical form of English sentences using *binary* quantifiers: quantifiers that take *two* open sentences and form a sentence:

[1] The noun can sometimes be omitted, when it is clear from context: cf. "Most went," "Few went," "Many went," "Two went," "Some went," and "All went." Note, however, that "No went" and "A went" always seem bad.

- $all_x(Fx, Gx)$ — All Fs are G.
- $at\text{-}least\text{-}two_x(Fx, Gx)$ — At least two Fs are G.
- $most_x(Fx, Gx)$ — Most Fs are G.

Why did we not do it this way from the start? Well, in a way, we did. Aristotelian logic and medieval logic used binary quantifiers. It was Frege, Russell, Peano, and the other mathematical logicians of the early twentieth century who got us using the unary quantifier. This was because it was convenient to do so: the semantics and proof theory are simpler, and the unary quantifiers are just what is needed to explicate mathematical concepts and facilitate mathematical proof. But in part it was a reaction against the earlier logical tradition, which Frege regarded as too slavishly wedded to traditional syntax. (See §3 of Frege's *Begriffsschrift*, where he rails against the subject-predicate analysis of the sentence.)

2 Semantics of binary quantifiers

The semantics of binary quantifiers is relatively straightforward. For comparison, here is how we define truth in a model on an assignment for formulas headed by a unary quantifier:

- $\vDash^a_{\mathcal{M}} \forall\alpha\psi$ iff for *every* assignment a' that agrees with a on the values of every variable except possibly α, $\vDash^{a'}_{\mathcal{M}} \psi$.
- $\vDash^a_{\mathcal{M}} \exists\alpha\psi$ iff for *some* assignment a' that agrees with a on the values of every variable except possibly α, $\vDash^{a'}_{\mathcal{M}} \psi$.

And here's how we'd do it for some binary quantifiers:

- $\vDash^a_{\mathcal{M}} all_\alpha(\phi, \psi)$ iff for *every* assignment a' such that $\vDash^{a'}_{\mathcal{M}} \phi$ and a' agrees with a on the values of every variable except possibly α, $\vDash^{a'}_{\mathcal{M}} \psi$.
- $\vDash^a_{\mathcal{M}} some_\alpha(\phi, \psi)$ iff for *some* assignment a' such that $\vDash^{a'}_{\mathcal{M}} \phi$ and a' agrees with a on the values of every variable except possibly α, $\vDash^{a'}_{\mathcal{M}} \psi$.
- $\vDash^a_{\mathcal{M}} most_\alpha(\phi, \psi)$ iff for *most* assignments a' such that $\vDash^{a'}_{\mathcal{M}} \phi$ and a' agrees with a on the values of every variable except possibly α, $\vDash^{a'}_{\mathcal{M}} \psi$.
- $\vDash^a_{\mathcal{M}} at\text{-}least\text{-}two_\alpha(\phi, \psi)$ iff for *at least two* assignments a' such that $\vDash^{a'}_{\mathcal{M}} \phi$ and a' agrees with a on the values of every variable except possibly α, $\vDash^{a'}_{\mathcal{M}} \psi$.

Whew! A mouthful. No wonder the early pioneers of logic didn't do it this way.

3 Defining binary quantifiers in terms of unary ones

Indeed, what Frege noticed is that one could *define* the traditional binary quantifiers used in syllogistic logic in terms of monadic quantifiers and truth-functional connectives. You already know how that can be done:

- $some_\alpha(\phi, \psi) \Leftrightarrow \exists x(\phi x \wedge \psi x)$
- $all_\alpha(\phi, \psi) \Leftrightarrow \forall x(\phi x \supset \psi x)$
- $at\text{-}least\text{-}two_\alpha(\phi, \psi) \Leftrightarrow \exists x \exists y(x \neq y \wedge \phi x \wedge \phi y \wedge \psi x \wedge \psi y)$
- $at\text{-}most\text{-}one_\alpha(\phi, \psi) \Leftrightarrow \forall x \forall y((\phi x \wedge \phi y \wedge \psi x \wedge \psi y) \supset x = y)$

4 Most—an essentially binary quantifier

Our success here might encourage us to think that this trick can always be pulled off: given *any* binary quantifier, we can define it in terms of truth-functional connectives and unary quantifiers. But it turns out that this is not the case. And the problem is not just that some binary quantifiers (like *a few*) are vague, and others (like *enough*) context-sensitive. There are perfectly precise binary quantifiers that cannot be defined in terms of unary quantifiers.

A paradigm example is *Most*, interpreted as meaning *more than half*. You might think, initially, that the binary quantifier *Most* could be defined in terms of a unary quantifier M, where $Mx\phi$ is true in a model just in case more objects in the domain satisfy ϕ than do not. But how? We might start by formalizing *Most cows eat grass* as $Mx(Cx \supset Gx)$, but this will be true in *any* model where cows make up fewer than half the objects in the domain, no matter how many of them eat grass. On the other hand, $Mx(Cx \wedge Gx)$ will be true *only* in models where cows are the majority of objects in the domain. So neither definition captures the meaning of *Most cows eat grass*. Of course, there are other things we could try. (Try them on your own, and convince yourself that nothing like this is going to work.[2])

5 Unary quantifiers beyond ∀ and ∃

We don't need to look to binary quantifiers to find quantifiers that resist definition in terms of ∃ and ∀. Try defining *M* (our unary quantifier "most objects in the domain") in terms of ∃ and ∀. There are other unary quantifiers that cannot be defined in terms of ∃ and ∀, including "there are finitely many," "there are infinitely many," and "there are an even number of." Adding these quantifiers to standard first-order logic yields more expressively powerful logics.[3] (There is a cost, though: these logics are usually harder to deal with than first-order logic, and various nice properties of first-order logic fail to hold when the new quantifiers are added.)

Exercise (extra credit):

5.1 There's no way to *say* "there are infinitely many Fs" in standard first-order logic. Still, one can write sentences of first-order logic that only have models with infinite domains. Can you come up with one?

6 Generalized quantifiers

Logicians and linguists have tried to generalize the notion of a quantifier in a precise way. This work is called the theory of *generalized quantifiers*.

Here is one common characterization, which will be suitable for our purposes:

Quantifier (1) An n-ary quantifier Q expresses a relation among n sets and the domain.

[2] The result was first proved in Barwise and Cooper, "Generalized Quantifiers and Natural Language," *Linguistics and Philosophy* 4 (1981), 159–219, Appendix C, C12 and C13.

[3] The bible for this kind of thing is Jon Barwise and Solomon Feferman's *Model-Theoretic Logics* (Berlin: Springer, 1985).

Sometimes an additional constraint is imposed, at least on *logical* quantifers: that the relation must be *topic-neutral*. A relation is topic-neutral if it does not depend on the particular individuals that belong to the sets.[4] The point of this restriction is to rule out, for example, a unary quantifier "everything except Jonathan." You can think of a topic-neutral relation between sets as a *purely quantitative* relation.

Let's think this through with some examples (where D is the domain):

- $\forall x \phi$ expresses the condition $\{x : \phi x\} = D$.
- $\exists x \phi$ expresses the condition $\{x : \phi x\} \cap D \neq \emptyset$.
- $all_x(\phi, \psi)$ expresses the condition $\{x : \phi x\} \supset \{x : \psi x\}$.
- $most_x(\phi, \psi)$ expresses the condition $|\{x : \phi x\} \cap \{x : \psi x\}| > \frac{1}{2}|\{x : \phi x\}|$, or equivalently $|\{x : \phi x\} \cap \{x : \psi x\}| > |\{x : \phi x\} \setminus \{x : \psi x\}|$.

Set-theoretic Notation:

- $\{x : \phi x\}$ — the set of things that satisfy ϕx.
- $|S|$ — the number of members S contains (its *cardinality*).
- $S \cap T$ — the *intersection* of S and T (the set of members belonging to both in common).
- $S \setminus T$ — the *difference* of S and T (the set containing all the members of S that are *not* in T).

In what follows I will sometimes use the abbreviation F for $\{x : Fx\}$ when it is clear that a set is intended.

7 Descriptions as quantifiers

A **definite description** is a phrase that denotes an object as the unique thing satisfying a certain description, for example, *the present king of France*, *the first dog born at sea*, *the bed*, *John's father* (=? *the father of John*), and perhaps also 2 + 6 (=? *the sum of* 2 *and* 6). Although not all definite descriptions have the form *the* ϕ, they can all be rephrased that way, so we'll talk in what follows as if all definite descriptions have that form.

There is no consensus among linguists and philosophers of language about the semantics, or even the grammar, of definite descriptions in natural languages. Definite descriptions are, in many ways, like *terms*: they can appear in many of the same places in a sentence as a name or pronoun:

John kicked *Sam*.	John kicked *the wall*.
Superman is *Clark Kent*.	Superman is *the nerdy reporter*.
Sam, Judy, and *Pete* went swimming.	Sam, Judy, and *the teacher* went swimming.

[4]This notion can be defined precisely in terms of invariance under permutations of the domain. We won't go into this here, but you may wish to consult Section 5 of my Stanford Encyclopedia of Philosophy article on logical constants.

Those who are impressed by this usually take definite descriptions to be *terms* grammatically, and *referring expressions* semantically. On the other hand, definite descriptions also pattern with quantifiers:

Two men kicked Sam.	*The* men kicked Sam.
Every dirty dog in the street barked.	*The* dirty dog in the street barked.
Every boy loves *some* girl.	Every boy loves *the* girl.

Those who are impressed by these similarities have taken definite descriptions to be *quantifiers*. Here I'll present the quantificational alternative. Later, when we're discussing the "slingshot argument," we'll consider how a non-quantificational account of definite descriptions might go.

We said that a binary quantifier can be thought of as expressing a relation among sets. For example, $\textit{at-least-two}_x(Fx, Gx)$ says that $|F \cap G| \geq 2$ (the set of elements common to F and G has two or more members), and $\textit{most}_x(Fx, Gx)$ says that $|F \cap G| > |F \backslash G|$ (the set of elements common to F and G has more members than the set of elements that belong to F but not G). Can we give a parallel treatment of *the*? Well, what must be the case if *The F is G* is to be true? Surely, there must *be* an F. And presumably there can't be more than one F. Finally, this F must be G. Taken together, these conditions are plausibly necessary and sufficient for the truth of *The F is G*. But clearly, these conditions can be represented as relations among sets. $\textit{the}_x(Fx, Gx)$ iff $|F| = |F \cap G| = 1$.

We can define truth in a model on an assigment for our new quantifier as follows.

the $\vDash^a_{\mathcal{M}} \textit{the}_\alpha(\phi, \psi)$ iff

- there is exactly one assignment a' such that a' differs from a at most in the value of α and $\vDash^{a'}_{\mathcal{M}} \phi$, and
- there is exactly one assignment a' such that a' differs from a at most in the value of α and $\vDash^{a'}_{\mathcal{M}} \phi$ and $\vDash^{a'}_{\mathcal{M}} \psi$.

So is *the* in English the quantifier $\textit{the}(,)$? As I mentioned, this is a highly contentious question. The match is pretty close. We don't use *the F* when there is more than one (salient) F, or when there aren't any.[5] So it is tempting to suppose that *the F is G* just *means* that there is a unique (salient) F and it is G. If that's right, then *the* in English is a quantifier.

However, the quantificational analysis also predicts that a sentence like *The present king of France is bald* should come out *false*. And that has seemed odd to many philosophers. Surely, they say, if there is no present king of France, then *The present king of France is bald* fails to make a determinate claim—and so fails to be either true *or* false. One who uses this sentence to make an assertion may *presuppose* that there is a present king of France, but it seems odd to say (with the quantificational account) that the sentence *entails* this.

8 Definite descriptions and scopes

One straightforward prediction of the quantificational account is that definite descriptions, like other quantifiers, have *scopes*. This means that certain English sentences will be predicted to have two readings, depending on how the scope ambiguity is resolved. Consider, for example:

(1) Fifteen presidential candidates are not campaigning in California.

This might mean either of two things:

(1w) $\textit{fifteen}_x(Px, \neg Cx)$
There are fifteen presidential candidates who are not campaigning in California.

[5] Why the qualification "salient"?

(1n) $\neg \mathit{fifteen}_x(Px, Cx)$
It's not the case that there are fifteen presidential candidates who are campaigning in California. (There are only six.)

In (1w), the quantifier takes *wide scope* over the negation. In (1n), it takes *narrow scope* (and the negation takes wide scope).

Do we see this phenomenon with definite descriptions? Consider:

(2) The present king of France is not washing my car.

(2w) $\mathit{the}_x(Kx, \neg Wx)$
The present king of France is such that: he is not washing my car.

(2n) $\neg \mathit{the}_x(Kx, Wx)$
It's not the case that the present king of France is washing my car.

On the quantificational reading, (2w), but not (2n), entails that there *is* a present king of France. Can we use (2) to mean both (2w) or (2n)? Or can we get only one reading? Ask yourself whether (2) can be *true* if (as is actually the case) there is no present king of France.

9 Russell's theory of descriptions

Historically, the quantificational account of definite descriptions is due to Bertrand Russell, while the nonquantificational approach was championed by Frege and Strawson.[6]

Definite descriptions as "incomplete symbols." Our approach so far has been to analyze *the* as a binary quantifier. Russell's actual approach was a bit different. He did not have the theory of generalized quantifiers at his disposal. So instead of representing *the* as a quantifier, as we did above, he represented *the F* as a kind of *term*, which he then showed how to eliminate in favor of (standard) quantifiers.

Russell's definite description terms are constructed using an upside-down iota (ι). ι is a variable-binding operator, just like $\forall$ and $\exists$, but unlike them it forms a *term*, not a *formula*. If ϕ is a formula and α is a variable, then $\iota\alpha\phi$ is a term. For example:

- the F = ιxFx
- the F that Gs b = $\iota x(Fx \wedge Gxb)$

Terms formed using ι can occur in formulas wherever other kinds of terms (variables and individual constants) can occur. For example:

- the F is H: $H\iota xFx$
- the F Gs the H that Gs the K: $G(\iota xFx)(\iota x(Hx \wedge Gx\iota yKy))$

[6] If you'd like to explore this debate, an excellent place to start is Gary Ostertag's anthology *Definite Descriptions: A Reader* (Cambridge: MIT Press, 1998). Stephen Neale's book *Descriptions* (Cambridge: MIT Press, 1990) is an influential presentation and defense of the quantificational view.

(Put parentheses around iota-terms when there is threat of ambiguity.)

Russell understood the terms formed using his upside-down iota not as genuine terms, but as "incomplete symbols." In effect, he took formulas containing iota-terms to be *abbreviations* for formulas not containing them.

It is not hard to convince yourself that, unlike *most*$(,)$, the binary quantifier *the*$(,)$ *can* be defined using standard first-order quantifiers:

(R1) $the_x(\phi x, \psi x) \equiv \exists x(\phi x \wedge \forall y(\phi y \supset y = x) \wedge \psi x)$

(The ϕ is ψ iff there is a unique ϕ and it is ψ.) This is essentially the equivalence Russell uses to eliminate definite descriptions, but there is a twist due to his use of $\imath$ terms rather than binary quantifiers. As a first attempt at translating (R1) to Russell's notation, we might try:

(R2) $\psi \imath x \phi x \equiv \exists x(\phi x \wedge \forall y(\phi y \supset y = x) \wedge \psi x)$.

However, there is a problem with (R2) as it stands. The problem stems from the fact that definite descriptions, like other quantifiers, have *scopes.* As noted above, we should get two different readings of "the present king of France is not washing my car." How can we represent these two distinct quantificational readings using the iota-term notation? As it stands, we can't. The formula

$$\neg R \imath x P x \tag{1}$$

is *ambiguous* between a narrow-scope and a wide-scope reading. (R2), as it is currently stated, says that it is equivalent to *both*

$$\neg \exists x(Px \wedge \forall y(Py \supset y = x) \wedge Rx) \tag{2}$$

(taking ψx to be Rx) *and*

$$\exists x(Px \wedge \forall y(Py \supset y = x) \wedge \neg Rx) \tag{3}$$

(taking ψx to be $\neg Rx$). But (1) can't be equivalent to both (2) and (3), because they aren't equivalent to each other! Our rule (R2) is not sound.

What we need to solve this problem is a way of indicating the scope of definite descriptions written using iota-terms. Russell and Whitehead do this in *Principia Mathematica* by putting a copy of the iota term in square brackets in front of the description's scope.[7] So, the narrow-scope reading of (1) would be written

$$\neg[\imath x P x] R \imath x P x \tag{4}$$

and the wide-scope reading would be written

$$[\imath x P x] \neg R \imath x P x. \tag{5}$$

(Note that the bracketed iota-term serves no function other than to indicate scope.) Using this notation, we can write a (sound) version of our equivalence rule:

(R3) $[\imath x \phi x] \psi \imath x \phi x \equiv \exists x(\phi x \wedge \forall y(\phi y \supset y = x) \wedge \psi x)$.

Following Russell and Whitehead, we will adopt the convention that if the scope-indicator is omitted, the iota-term will be assumed to have the narrowest possible scope. Thus,

$$\neg R \imath x P x \tag{6}$$

is to be read as

$$\neg[\imath x P x] R \imath x P x, \tag{7}$$

which according to (R3) is equivalent to

$$\neg \exists x(Px \wedge \forall y(Py \supset y = x) \wedge Rx). \tag{8}$$

[7] See Stephen Neale, *Facing Facts*, 95–6.

10 Proofs

Since for any formula containing the quantifier *the* or Russell's ι operator we can always find an equivalent formula that uses only the standard quantifiers, it is easy to extend our proof system to accommodate definite descriptions.

Russellian Equivalences: Whenever you have the right-hand side of an instance of (R1) or (R3), you may replace it with the left-hand side, and vice versa, with justification "RE." This is a substitution rule, so it may be used even on subformulas.

Examples:

1	$the_x((Fx \wedge Gx), Hx)$	hyp.
2	$\exists x((Fx \wedge Gx) \wedge \forall y((Fy \wedge Gy) \supset y = x) \wedge Hx)$	RE 1 ($\phi x = Fx \wedge Gx$, $\psi x = Hx$)

1	$\exists x(Gx \wedge \forall y(Gy \supset y = x) \wedge (Fx \supset Hx))$	hyp.
2	$[\iota xGx](F\iota xGx \supset H\iota xGx)$	RE 1 ($\phi x = Gx$, $\psi x = Fx \supset Hx$)

Important note: Although terms formed using ι are grammatically terms, they do not function *semantically* as terms (on Russell's account). Thus

- In specifying a model, you do *not* specify an interpretation for these terms.
- You cannot use these terms to get substitution instances when doing $\forall$ Elim, $\forall$ Intro, $\exists$ Elim, or $\exists$ Intro.[8]
- In particular, you cannot instantiate $\forall x(x = x)$ with ιxFx to get $\iota xFx = \iota xFx$. You'd better not be able to, because $\iota xFx = \iota xFx$ implies $\exists xFx$. So you'd be able to prove the existence of an F for any F! (Contrary to what you may be thinking, this is not a good thing.)

Exercises:

10.1 How would you express the following sentences in logical notation? Do it first using the generalized quantifier $the(,)$, and then using the Russellian ι operator.

(a) The man who killed Kennedy is a murderer.

(b) The shortest spy is the tallest pilot.

(c) Not every woman likes her father.

10.2 Give a deduction of $\exists xFx$ from $[\iota xFx](\iota xFx = \iota xFx)$.

10.3 Show that the = Elim rule is still valid when one term has the form ιxFx, not just when both terms are individual constants. That is, give a deduction that shows the validity of the following:

$$a = \iota xFx,\ Ga,\ /\ G\iota xFx$$

[8] This may seem too restrictive. After all, if we have *the farthest star is a gas giant*, can't we conclude *something is a gas giant*? Yes—but you can get this conclusion even with the restrictive rules we have. (Convince yourself of this.)