Identity

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1 Grammar

The identity sign ('=') is a two-place predicate. By convention, we write one argument on the left and one on the right (as in '$a = b$'). We should not let this convention obscure the fact that grammatically '=' is just a two-place predicate, like the 'G' in Gab. We could just as well have written $= ab$ or Iab.

2 Semantics

The extension of '=' in a model is the relation each thing bears to itself and to no other thing (the identity relation). For example, if the domain is $\{1, 2, 3\}$, then the extension of '=' is $\{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle\}$. That is not to say that all true identity statements are the tautologous kind ($a = a$). $a = b$ can be true in a model, provided that a and b get assigned the same interpretation (the same object) in that model.

3 Proofs

To do proofs with identity, we'll need two new rules.

= Intro Where α is an individual constant, you may write $\alpha = \alpha$ on any line of a proof, with justification "= Intro." (Sometimes this rule is also called "Reflexivity.")

= Elim From premises $\alpha = \beta$ (or $\beta = \alpha$) and $\phi(\alpha)$, you may conclude $\phi(\beta)$, with justification "= Elim." Here α and β are individual constants, $\phi(\alpha)$ is the result of substituting α for x in an open formula $\phi(x)$, and $\phi(\beta)$ is the result of substituting β for x in the same open formula. (Sometimes this rule is called "Substitution of Identicals" or "Indiscernibility of Identicals.")

Note that *both* = Elim steps in the following proof are valid:

1	$\exists x Raxa$	
2	$a = b$	
3	$\exists x Rbxb$	= Elim, 1, 2
4	$\exists x Rbxa$	= Elim, 1, 2

(1)

Line 3 is a valid step because line 1 is the result of substituting a for y in the open formula $\exists x Ryxy$, and line 3 is the result of substituting b for y in this same open formula. Line 4 is a valid step

because line 1 is the result of substituting a for y in the open formula $\exists x Ryxa$, and line 4 is the result of substituting b for y in this same open formula.

This is all you need for proofs with identity. Here's an example.

1	$\exists x(Fx \wedge Gxb)$	
2	$a = b$	
3	$\exists x(Fx \wedge Gxa)$	= Elim, 1, 2
4	$Fc \wedge Gca$ $\boxed{\text{c}}$	3 c/x
5	$c = c$	= Intro
6	$Gca \wedge c = c$	Taut Con 4, 6
7	$\exists x(Gxa \wedge x = x)$	$\exists$ Intro 7 c/x
8	$\exists x(Gxa \wedge x = x)$	$\exists$ Elim, 3, 4–7

(2)

Exercises:

1. How would you express the following in predicate logic with identity?

   (a) Every logician loves someone other than herself.

   (b) The only one who respects Richard is Sue.

   (c) There are at least two rich dogs.

   (d) There are at most two smart dogs.

   (e) Liz is the tallest spy.

   (f) Liz is the tallest rider who roped at least two calves.

2. (a) Give a formula, using quantifiers and identity, that is true in every model with a domain of one object and false in some model with a domain of two objects. (b) Give a formula, *not* using quantifiers and identity, that has this property.

3. Prove that the following rules are valid. (Give a deduction.) Once you have done this, you may use these derived rules to simplify proofs with identity.

   Symmetry

   $$\frac{a = b}{b = a}$$

   Transitivity

   $$\frac{a = b \qquad b = c}{a = c}$$

4. Prove $Fa \equiv \exists x(Fx \wedge x = a)$.

5. Suppose that you have a quantifier $\exists^n x$, meaning "there are at least n x..." How could you define $\exists^{n+1} x$ in terms of $\exists^n x$?

6. The identity sign is treated differently from other predicates in first-order logic. Can you think of any reasons for this?