Prawitz's proof-theoretic account of consequence

John MacFarlane — Philosophy 142

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1 Definitions

An **argument**, for Prawitz, is a step-by-step deduction, not a pair of premises and a conclusion. (A **proof** is an abstract entity that is "expressed" by an argument.) Prawitz uses Gentzen's "tree-style" natural deductions, rather than the Fitch-style deductions we're used to.

An argument is **closed** if it has no assumptions (unbracketed premises—bracketed ones are "hypotheses") and no unbound variables. Otherwise, it is **open**. The argument on the left below is open, because it has assumptions A_1 and A_2. The argument on the right is closed, because it has no assumptions or unbound variables.

$$\frac{A \qquad B}{A \wedge B} \qquad\qquad \frac{\begin{matrix}[A] \\ \Delta \\ B\end{matrix}}{A \rightarrow B}$$

Here Δ stands for a valid argument from A to B.

An argument is **canonical** iff it ends with an introduction rule and contains valid (open or closed) arguments for the premises.

What is it for an argument to be **valid**?

- An *open argument* is valid iff the result of replacing each assumption with a valid closed argument for that assumption is valid.
- A *closed* argument is valid iff either (a) it is a canonical argument or (b) it can be reduced to a canonical argument for its conclusion.

This looks circular, since canonical argument is defined in terms of valid argument, and valid argument is defined in terms of canonical argument. Actually, it's not, because to settle whether an argument for a conclusion is canonically valid, one need only settle the validity of arguments for less complex conclusions. This means that the process will eventually end, and not go on forever. So canonical validity and validity can be defined by mutual recursion.

2 Reduction

What is it for an argument to be **reducible** to a canonical argument for its conclusion? There must be an effective procedure for transforming the one into the other. This is most easily explained through some examples.

2.1 ∧ Elim

Let's first look at how we might show that conjunction introduction and elimination rules are valid.

$$\frac{A \qquad B}{A \wedge B}$$

∧ Intro

$$\frac{A \wedge B}{A}$$

∧ Elim

These are both open arguments, since they have assumptions. So, to show that they are valid is to show that the closed argument you get when you replace each assumption with a valid closed argument for that assumption is valid. Letting $\Delta_1, \Delta_2, \Delta_3$ stand for valid closed arguments, then, we need to show that all instances of the following are valid:

$$\frac{\begin{array}{cc}\Delta_1 & \Delta_2 \\ A & B\end{array}}{A \wedge B}$$

(Closed) ∧ Intro

$$\frac{\begin{array}{c}\Delta_3 \\ A \wedge B\end{array}}{A}$$

(Closed) ∧ Elim

Now, it's trivial to show that instances of the argument form on the left (above) are valid. Any such instance will be a canonical argument, because it ends with the use of an introduction rule and contains valid arguments for the premises of the rule. For Prawitz, introduction rules need no further justification. They have the status of "definitions" of the logical constants they introduce.

More work is required to show that instances of the argument on the right is valid, since they are not canonical arguments (they don't end with introduction rules). To show that such an instance is valid, we need to show that it can be transformed into a canonical argument.

How do we do this? Well, Δ_3 is a valid argument for $A \wedge B$. So it must either be a canonical argument or be reducible to one. A canonical argument for $A \wedge B$ is one that ends in an application of the ∧ Intro rule. So from Δ_3 we can construct valid arguments Δ_4, Δ_5 such that

$$\frac{\begin{array}{cc}\Delta_4 & \Delta_5 \\ A & B\end{array}}{A \wedge B}$$

But now we can construct a canonical argument for the A, the conclusion of the ∧ Elim step. For this is a valid argument for A:

$$\begin{array}{c}\Delta_4 \\ A\end{array}$$

And because it is valid, it is either canonical or reducible to a canonical argument. Either way, we end up with a canonical argument for A.

What have we done? We've shown that the ∧ Elim rule is, in a sense, dispensible. Whenever there is a valid closed argument that uses ∧ Elim to get A, we can extract from it a valid closed argument that gets A without using ∧ Elim. In this way we can justify ∧ Elim purely proof-theoretically!

2.2 $\vee$ Elim

Here are the rules for $\vee$ introduction and elimination:

$$\frac{A}{A \vee B}\ \vee\text{ Intro} \qquad\qquad \frac{B}{A \vee B}\ \vee\text{ Intro} \qquad\qquad \frac{A \vee B \qquad \begin{matrix}[A]\\ C\end{matrix} \qquad \begin{matrix}[B]\\ C\end{matrix}}{C}\ \vee\text{ Elim}$$

To show that $\vee$ Elim is valid, we need to show that any valid closed argument for its premise $A \vee B$ can be reduced to a canonical argument for its conclusion C.

So, let Δ be a valid closed argument for $A \vee B$.

$$\frac{\begin{matrix}\Delta\\ A \vee B\end{matrix} \qquad \begin{matrix}[A]\\ C\end{matrix} \qquad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

Since Δ is valid, it follows (from the definition of “valid closed argument” above) that Δ can be reduced to a canonical argument—that is, an argument for $A \vee B$ that ends with an application of $\vee$ Intro and has valid arguments for the premises of the $\vee$ Intro step. We can get $A \vee B$ using $\vee$ Intro either from A or from B, so there are two possibilites here:

$$\frac{\begin{matrix}\Delta_3\\ A\\ A \vee B\end{matrix} \qquad \begin{matrix}[A]\\ \Delta_1\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \Delta_2\\ C\end{matrix}}{C} \qquad\qquad\qquad \frac{\begin{matrix}\Delta_4\\ B\\ A \vee B\end{matrix} \qquad \begin{matrix}[A]\\ \Delta_1\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \Delta_2\\ C\end{matrix}}{C}$$

But now we can rearrange our proofs as follows:

$$\begin{matrix}\Delta_3\\ A\\ \Delta_1\\ C\end{matrix} \qquad\qquad\qquad\qquad \begin{matrix}\Delta_4\\ B\\ \Delta_2\\ C\end{matrix}$$

Both of these are valid closed arguments for C, so (by the definition of valid closed argument) they can be reduced to canonical arguments for C, and we’re done.

3 Philosophical reflections

What has Prawitz done? He has given an account of what it is for an argument to be valid that does not appeal to truth, satisfaction, or other semantic notions. It is also immune to Tarski’s objections based on Gödel’s incompleteness theorems, because it does not identify validity with derivability in any *particular* formal system. So Gödel’s demonstration that no one formal system can capture all the valid arguments in a (sufficiently powerful) language need not bother us.

To motivate his account, Prawitz offers a nice analogy.[1] Consider expressions that purport to denote natural numbers, for example:

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[1] “Remarks on Some Approaches to the Concept of Logical Consequence,” *Synthese* 62 (1985), 153–171.

- 1678
- $(3 + 6) - 15$
- the largest even number less than 11
- the largest even number

Some of these expressions do denote natural numbers, and others do not. How do we decide which do and which do not? For the first two on the list, it's trivial. You can tell just from the expression that there is such a natural number. They're in "canonical form." But for the others, we're not convinced that there is such a number until we are convinced that we can reduce it to canonical form. So, "what is the largest even number less than 11?" can be answered with "10", which is in canonical form. But there is no numeral n that answers the question "what is the largest even number?" And because of that, we say that there is no largest even number. So we can think of questions about whether certain expressions denote natural numbers as questions about whether these expressions can be reduced to a canonical form.

Prawitz's approach to the validity of arguments is similar. Certain closed arguments are *trivially* valid. These are the ones consisting entirely of introduction rules, which are trivially valid because "introduction inferences determine the meanings of the logical constants concerned" (685). Other arguments are said to be valid just in case there is a procedure for reducing them to trivially valid ones.

Note that Prawitz's definition depends on a substantive assumption (not discussed in the paper, though Dummett discusses it): the assumption that if a formula can be proved it all, a proof can always be given in canonical form. This is not so obvious. Couldn't we have primitive proof rules (not, of course, logical ones) that take us *directly* to $A \vee B$ without passing through either disjunct? Dummett's example: I can tell that either a girl is in the garden or a boy is in the garden without being able to tell which disjunct is true. So it seems I can verify this disjunction in a way that cannot be reduced to an application of the disjunction introduction rule. Perhaps the assumption is more plausible for mathematical discourse than for empirical discourse.

4 Intuitionistic logic

Prawitz's account of logical consequence is not only a philosophical alternative to Tarski's. It is also a *logical* alternative: it leads to a different answer to the question, "what follows from what?"

In classical logic, the following argument is valid:

$$\frac{}{A \vee \neg A}$$

This is a closed argument, because it has no assumptions or free variables. So, on Prawitz's criterion, it is valid just in case it is (a) canonical or (b) reducible to a canonical argument. It isn't canonical, since a canonical argument for $A \vee \neg A$ would have to proceed via $\vee$ Intro. So we must ask, is there an effective procedure for reducing it to a canonical argument?

Such an argument would look like one of the following:

$$\frac{\begin{array}{c}\Delta_1\\ A\end{array}}{A \vee \neg A} \qquad\qquad \frac{\begin{array}{c}\Delta_2\\ \neg A\end{array}}{A \vee \neg A}$$

But we have no ingredients in our original argument from which to construct an argument for either A or $\neg A$. So there isn't going to be a way to reduce our argument to one of these. So the argument isn't valid, on Prawitz's criterion. It *is* valid on the classical criterion—every classical model makes the conclusion true. So here is a real, logical difference between these two accounts of validity. Prawitz's criterion yields a logic called *intuitionistic logic*, not classical logic.

Another argument form that is classically, but not intuitionistically valid is double-negation elimination:

$$\frac{\neg\neg A}{A}$$

In Prawitz's system, $\neg A$ is defined as $A \to \bot$, so this becomes

$$\frac{(A \to \bot) \to \bot}{A}$$

Suppose we have a valid closed argument for the premise. Can we extract a canonical argument for the conclusion, A? Well, suppose we have a canonical argument for the premise:

$$\frac{\begin{array}{c}[A \to \bot] \\ \Delta \\ \bot\end{array}}{(A \to \bot) \to \bot}$$

This doesn't give us the ingredients to construct a canonical argument for A. So, Prawitz rejects double-negation elimination (and with it reductio proofs that end with double-negation elimination).

One might ask: why not take double-negation elimination *together with* the introduction rules as defining $\neg$? But remember that the basis of Prawitz's reply to Prior is that only introduction rules can count as definitions of the connectives they introduce. Once we allow both introduction and elimination rules to count as definitions, we open ourselves to the possibility of tonkish connectives.

Why the name "intuitionistic"? Intuitionistic logic originated from the intuitionist school in the philosophy of mathematics, which held that mathematics is about mental constructions ("intuitive" in roughly Kant's sense), not a mind-independent realm of platonic entities. On the intuitionistic conception, a mathematical object exists if it is possible to construct it. So, on the intuitionistic conception, one cannot demonstrate $\exists x \neg Fx$ by reducing to absurdity $\forall x Fx$: to show that $\exists x \neg Fx$ one actually needs to construct an instance $\neg Fa$, and it could be that, despite the absurdity of $\forall x Fx$, no such instance is constructible. So intuitionistic mathematics refuses to allow certain forms of proof that are allowed in classical mathematics.

One can think of the intuitionists as identifying truth with provability (or, as Prawitz puts it, the "potential existence of evidence," p. 681). This makes a certain amount of sense in mathematics. Suppose there is a statement that is neither provable nor refutable. On the Platonist conception of mathematics, there is nonetheless a fact of the matter about whether it is true or false—a fact that is completely inaccesible to human investigators. For intuitionists, by contrast,

> ... an understanding of a mathematical statement consists in the capacity to recognize a proof of it when presented with one; and the truth of such a statement can consist only in the existence of such a proof. ... Thus, while, to a platonist, a mathematical theory relates to some external realm of abstract objects, to an intuitionist it relates to our own mental operations: mathematical objects themselves are mental constructions, that is, objects of thought not merely in the sense that they are thought about, but in the sense that, for them, *esse est concipi*. They exist only in virtue of our mathematical activity, which consists in mental operations, and have only those properties which they can be recognized by us as having. (Michael Dummett, *Elements of Intuitionism*, pp. 4–5)

If truth is identified with provability, and falsity with refutability, then acceptance of the law of excluded middle ($A \vee \neg A$) amounts to acceptance of the claim that every statement is either provable or refutable. That is why intuitionists reject excluded middle. (For a nice introduction to intuitonisim, see the article on "Intuitionistic Logic" in the Stanford Encyclopedia of Philosophy.)

Exercises:

1. Intuitionists accept the law of non-contradiction, $\neg(A \wedge \neg A)$. How can they consistently accept this while rejecting the law of excluded middle, $A \vee \neg A$? Aren't these two laws equivalent, given De Morgan's laws?

2. Explain why Prior's elimination rule for *tonk* is not valid, according to Prawitz's definition of validity.

3. State an elimination rule that would be make sense, given the introduction rule for *tonk*, and show that it is valid, according to Prawitz's definition.