Relevance Logic

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1 The Lewis Argument

(1)	$P \wedge \neg P$	
(2)	P	(1, $\wedge$ Elim)
(3)	$\neg P$	(1, $\wedge$ Elim)
(4)	$P \vee Q$	(2, $\vee$ Intro)
(5)	Q	(3, 4, Disjunctive Syllogism)

This argument appears in C. I. Lewis and C. H. Langford, *Symbolic Logic*, second edition (New York: Dover, 1959), pp. 248–51. Anderson and Belnap note that the argument was known to Albert of Saxony and other medieval logicians.

2 Options

No one wants to reject $\wedge$ elimination. These options have all been tried:

(a) Reject $\vee$ Intro (a.k.a. "disjunctive weakening")

(b) Reject the transitivity of entailment

(c) Reject Disjunctive Syllogism

3 Rejecting disjunctive weakening

How can relevance be represented formally? *Variable sharing.*

Parry, "analytic implication": the conclusion may not contain any variables not contained in the premises. Compare Kant's definition of an analytic judgment as one in which the predicate is *contained in* the subject.

Note that P does not analytically imply $P \vee Q$.

Disadvantages:

- Disjunctive weakening is an absolutely crucial rule!
- The "containment" explication of analyticity forces us to reject disjunctive definitions. E.g. if "all husbands are spouses" is to be analytic, we must define "husband" as "male spouse" rather than defining "spouse" as "husband or wife."
- What could be more relevant to $P \vee Q$ than P?

Parry's view is discussed in Alan Ross Anderson and Nuel Belnap, *Entailment*, vol. 1 (Princeton University Press, 1975), sec. 29.6.1.

4 Rejecting transitivity

Motivation: In every single step of the Lewis argument, the premises seem relevant to the conclusion. It is only when we chain these together and conclude that (1) entails (5) that we get problems. So, maybe,

- (1) entails (2)
- (1) entails (3)
- (2) entails (4)
- (3) and (4) together entail (5)
- But (1) does not entail (5)

Can we describe a system that works this way? We want to exclude arguments with contradictory premises or tautologous conclusions. But not *all* such arguments! A relevantist still wants $P \wedge \neg P$ to entail $\neg P$, for example. But perhaps this is okay only because it is a substitution instance of a valid argument without a contradictory premise, namely: $P \wedge Q$, therefore Q.

Proposal: An argument is *valid* iff it is a substitution instance of an argument that

- is classically valid,
- does not have a contradictory premise, and
- does not have a tautologous conclusion.

Advantages:

- allows that each step of the Lewis argument is valid
- rejects very little classical reasoning (indeed, Neil Tennant has proved that in almost every case where there is a non-relevant proof, there will be a relevant one)

Disadvantages:

- What about the argument from $(P \wedge \neg P) \vee R$ to $Q \vee R$? This is classically valid, the premise is not contradictory, and the conclusion is not tautologous. Yet it seems objectionable from a relevance point of view. (Consider how you'd prove it.)
- Transitivity of entailment seems fundamental!

> Any criterion according to which entailment is not transitive, is *ipso facto* wrong. It seems in fact incredible that anyone should admit that B follows from A, and that C follows from B, but feel that some further argument was required to establish that A entails C. (Anderson and Belnap, *Entailment* vol. 1, p. 154)

The idea of rejecting transitivity was first proposed by Timothy Smiley in "Entailment and Deducibility," *Proceedings of the Aristotelian Society* n.s. 59 (1959), 233–54.

5 Rejecting Disjunctive Syllogism

We'll focus on Belnap and Anderson's logic of first-degree entailment (called $\mathbf{E_{fde}}$), which is equivalent to a logic devised by Ackermann in the 1930's.

Because we can always combine (a finite number of) premises into a single conjunction, we'll consider here only one-premise arguments, which we'll call "entailments." The goal will be to define a notion of "tautological entailment" that captures just the ones that are relevantly valid in virtue of their propositional forms.

An *atom* is a propositional constant or its negation:

- atoms: P, $\neg P$, Q
- not: $P \vee Q$, $P \wedge \neg R$

A *primitive conjunction* is a conjunction of atoms. A *primitive disjunction* is a disjunction of atoms.

- primitive conjunctions: $P \wedge \neg P \wedge Q$, $P \wedge Q \wedge R \wedge \neg S$
- primitive disjunctions: $P \vee Q \vee \neg P$, $P \vee R$
- neither: $(P \wedge Q) \vee R$, $Q \wedge (P \vee \neg R)$

$\phi \Rightarrow \psi$ is a *primitive entailment* if ϕ is a primitive conjunction and ψ a primitive disjunction.

- example: $P \wedge \neg P \wedge Q \Rightarrow P \vee R$
- not: $P \wedge \neg P \wedge Q \Rightarrow P \vee (R \wedge S)$

A primitive entailment $\phi \Rightarrow \psi$ is *explicitly tautological* if some (conjoined) atom of ϕ is identical with some (disjoined) atom of ψ.

- examples: $P \wedge \neg P \wedge Q \Rightarrow P \vee R$, $\neg P \wedge \neg Q \wedge \neg R \Rightarrow S \vee \neg Q$
- not: $P \wedge \neg P \Rightarrow Q$

This captures a certain kind of "containment" of conclusion in premises.

What about entailments that are *not* primitive entailments, like $P \vee Q \Rightarrow P \vee \neg(R \wedge \neg R)$? How do we test them?

1. Put the premise into *disjunctive normal form* (i.e., convert it into a disjunction of primitive conjunctions) and the conclusion into *conjunctive normal form* (i.e., a conjunction of primitive disjunctions). You should now have something of the form $\phi_1 \vee \phi_2 \vee \ldots \vee \phi_n \Rightarrow \psi_1 \wedge \psi_2 \wedge \ldots \wedge \psi_n$, where each ϕ_i is a primitive conjunction and each ψ_i is a primitive disjunction.
2. The entailment is a *tautological entailment* iff for every ϕ_i and ψ_j, $\phi_i \Rightarrow \psi_j$ is explicitly tautological.

This procedure depends on the fact that every entailment can be put into normal form using relevantly acceptable rules. Here is the algorithm:

1. Apply DeMorgan's laws and Double Negation Elimination to drive negations inward as far as possible.

2. Use Distribution and Commutation to move all disjunction signs outside conjunctions (for disjunctive normal form) or inside conjunctions (for conjunctive normal form).

3. Use Association to group things to the left. (Remember, $\psi_1 \wedge \psi_2 \wedge \psi_3$ is short for $(\psi_1 \wedge \psi_2) \wedge \psi_3$.)

This can always be done, and although there is not a unique normal form for each entailment, it can be proved that if one normal form passes the test, they all do.

Let's try a simple example:

$$P \vee Q \Rightarrow P \vee \neg(R \wedge \neg R)$$

The premise is already in DNF so we can leave it.

We need to put the conclusion in CNF. First, drive in the negation:

$$P \vee (\neg R \vee \neg\neg R)$$

$$P \vee (\neg R \vee R)$$

Now associate:

$$(P \vee \neg R) \vee R$$

which is

$$P \vee \neg R \vee R$$

We're there! (Note: it's a conjunction of a single disjunction.)

Now we ask:

- Is $P \Rightarrow P \vee \neg R \vee R$ explicitly tautological? YES
- Is $Q \Rightarrow P \vee \neg R \vee R$ explicitly tautological? NO

So it's not a tautological entailment. Try disjunctive syllogism!

Exercise:

1. Using De Morgan's laws, Double Negation Elimination, Commutation, Association, and Distribution, put the following sentences into *both* disjunctive normal form and conjunctive normal form. Show your work.

   (a) $(P \wedge Q) \vee (P \wedge \neg Q)$

   (b) $P \wedge Q \wedge \neg Q$

   (c) $\neg(P \vee Q) \wedge \neg(P \vee \neg Q)$

2. Use the procedure described above to determine whether the following classical entailments are tautological entailments:

   (a) $P \wedge \neg Q \Rightarrow P$

   (b) $P \Rightarrow (P \wedge Q) \vee (P \wedge \neg Q)$

   (c) $\neg(P \vee \neg Q) \Rightarrow \neg(P \vee (P \wedge R))$

Note that Disjunctive Syllogism is essentially *Modus Ponens* for the material conditional, since $\phi \supset \psi$ is equivalent to $\neg\phi \vee \psi$. Relevantists do not think the material conditional is a real conditional at all, so this is not a problem for them.

Advantages:

- This is the only solution if you want to keep disjunctive weakening and transitivity.

Disadvantages:

- There are fewer valid arguments than with the "rejecting transitivity" approach, so this approach is more revisionary.
- It is hard to see intuitively how there is not a relevant connection between premises and conclusion in disjunctive syllogism.

This approach is developed in Anderson and Belnap, *Entailment*, vol. 1, sec. 15.1 (in your reader).

6 Four-valued tables

It turns out that the logic of tautological entailment can be captured using four-valued truth tables. The four truth values are sets of regular truth values: $\{\mathbf{T}\}$, $\{\mathbf{F}\}$, $\{\}$, $\{\mathbf{T},\mathbf{F}\}$. Here are the truth tables for $\neg$ and $\wedge$:

	$\{\}$	$\{\mathbf{F}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T},\mathbf{F}\}$
$\neg$	$\{\}$	$\{\mathbf{T}\}$	$\{\mathbf{F}\}$	$\{\mathbf{T},\mathbf{F}\}$

$\wedge$	$\{\}$	$\{\mathbf{F}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T},\mathbf{F}\}$
$\{\}$	$\{\}$	$\{\mathbf{F}\}$	$\{\}$	$\{\mathbf{F}\}$
$\{\mathbf{F}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$
$\{\mathbf{T}\}$	$\{\}$	$\{\mathbf{F}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T},\mathbf{F}\}$
$\{\mathbf{T},\mathbf{F}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$	$\{\mathbf{T},\mathbf{F}\}$	$\{\mathbf{T},\mathbf{F}\}$

$\phi \Rightarrow \psi$ is a tautological entailment iff any assignment of values to propositional constants that makes ϕ *at least* **T** makes ψ *at least* **T**, and any assignment of values to propositional constants that makes ψ *at least* **F** makes ϕ *at least* **F**. (The values $\{\mathbf{T}\}$ and $\{\mathbf{T},\mathbf{F}\}$ are at least **T**, and $\{\mathbf{F}\}$ and $\{\mathbf{T},\mathbf{F}\}$ are at least **F**.)

Exercise:

3. What should the table for $\vee$ look like? Study the table for $\wedge$ and figure out the principles behind its construction, and apply these to $\vee$.
4. Use your tables, and the definition of tautological entailment above, to test (2a) and (2b) for tautological entailment. You don't need to give the whole truth table (which can be pretty large with a four-valued logic), but be sure to show your work.