Natural Deduction for Relevance Logics*

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Last time we looked at a system for assessing "first-degree entailments," which are so-called because they contain only one occurrence of the entailment arrow. But, supposing we think of this as a kind of conditional, it should be able to nest. Can we do logic for a conditional that expresses entailment?

One way to get a feel for this is to look at natural deduction systems. (But for simplicity, we'll consider systems with just the conditional →, no other operators.) Let's first consider how we can get "paradoxes of material implication" using standard rules:

1	│ P (hypothesis)		
2	│ │ Q (hypothesis)		
3	│ │ P	Reit	(1)
4	│ $Q \rightarrow P$	→ Intro 2–3	
5	$P \rightarrow (Q \rightarrow P)$	→ Intro 1–4	

This is "irrelevant," and relevance logicians want to reject it.

One way to reject it is to restrict the Reit move in line 3. We've seen a similar strategy in the natural deduction rules for modal operators, which use subproofs restricting reiteration. We can get the same effect here by allowing only conditionals to be reiterated into → Intro subproofs. The resulting system is $\mathbf{S4}_{\rightarrow}$ (the conditional fragment of **S4**, in which $\phi \rightarrow \psi$ can be defined as $\Box(\phi \supset \phi)$). But this still allows us to get some "irrelevant" theorems, for example:

1	│ $P \rightarrow R$ (hypothesis)		
2	│ │ Q (hypothesis)		
3	│ │ $P \rightarrow R$	Reit	(2)
4	│ $Q \rightarrow (P \rightarrow R)$	→ Intro 2–3	
5	$(P \rightarrow R) \rightarrow (Q \rightarrow (P \rightarrow R))$	→ Intro 1–4	

The relevance logic **R** blocks the original proof differently. According to **R**, there's nothing wrong with reiterating P into the subproof. The problem, rather, is that the hypothesis, Q, is not *used*. We fix this by requiring that the hypothesis be used.

Formally, this is done by subscripting. Each formula has, as subscript, a set of numerical indices. When a formula is reiterated, its indices are carried along with it. When a rule uses two or more

*I am indebted in my presentation to seminars given by Nuel Belnap. For a presentation that closely parallels this one, see the first five sections of Anderson and Belnap, *Entailment* vol. 1 (Princeton: Princeton University Press, 1975).

premises, all of the indices are combined (we take the union of the sets of indices for the premises). When a hypothesis is introduced, it comes with a new index of its own (a singleton set). When a hypothesis is discharged (→ Intro), its index is subtracted from the index of the consequent, *in which it must be included.* (This is how we require that the hypothesis actually be used.)

So let's see how our proof is blocked:

$$
\begin{array}{lll}
1 & \quad P_{\{1\}} & \\
2 & \quad\quad Q_{\{2\}} & \\
3 & \quad\quad P_{\{1\}} & \text{Reit} \\
4 & \quad Q \to P & \text{Illegal! } 2 \notin \{1\} \\
5 & P \to (Q \to P) &
\end{array}
\tag{3}
$$

Proof (2) is blocked for similar reasons. But there are still some things we get that we don't want. For example, the "law of assertion:" $A \to ((A \to B) \to B)$:

$$
\begin{array}{lll}
1 & \quad A_{\{1\}} & \\
2 & \quad\quad A \to B_{\{2\}} & \\
3 & \quad\quad A_{\{1\}} & \text{Reit} \\
4 & \quad\quad B_{\{1,2\}} & \to \text{Elim, 2, 3} \\
5 & \quad (A \to B) \to B_{\{1\}} & \to \text{Intro, 2–4} \\
6 & A \to ((A \to B) \to B)_{\{\}} & \to \text{Intro, 1–5}
\end{array}
\tag{4}
$$

What's wrong with this law, from a relevance point of view? Intuitively: the truth of A does not seem relevant to whether $(A \to B)$ implies/entails B.

This is not a problem unless we want our conditional to *express* (rather than merely indicating) entailment. (Here I'm using terminology from Meyer's article "Entailment.") For it is only in that case that we can read $A \to B$ as "A entails B."

To get a conditional that expresses entailment (and this is system **E**), we need to impose *both* the **S4** restriction on Reit *and* the indexing system of **R**. Then the proof is blocked. So, **E** = **S4** + **R**.

You might conjecture that a sentence is a theorem of $\mathbf{E}_{\to}$ just in case it is a theorem of $\mathbf{S4}_{\to}$ *and* of $\mathbf{R}_{\to}$. But this isn't the case! Kripke came up with a countexample: $A \to ((A \to (A \to B)) \to (A \to B))$ is not a theorem of $\mathbf{E}_{\to}$, but it is a theorem of both $\mathbf{R}_{\to}$ and $\mathbf{S4}_{\to}$.

Try proving it in all three systems...

$$
\begin{array}{lll}
1 & \quad A_{\{1\}} & \\
2 & \quad\quad A \to (A \to B)_{\{2\}} & \\
3 & \quad\quad ? & \longleftarrow \text{ can't reit } A\text{, so can't finish} \\
4 & \quad\quad ? & \text{In } \mathbf{R}\text{, we can reit } A\text{; in } \mathbf{S4}\text{, we don't need to use it} \\
5 & \quad\quad (A \to B) & \text{in this subproof.} \\
6 & \quad (A \to (A \to B)) \to (A \to B) & ? \\
7 & A \to ((A \to (A \to B)) \to (A \to B)) & ?
\end{array}
\tag{5}
$$